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MAXIMAL LINEAR TOPOLOGIES AND THE COMPLEMENT OF LINEAR TOPOLOGIES

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Introduction. The purpose of this paper is two-fold. First, we characterize the maximal left linear topology on a ring. Applying this, we again prove the equivalence of conditions on a ring, obtained by Nicholson and Sarath [6], to have a unique maximal left linear topology. Secondly, as in parallel with Meijer and Smith [7], we investigate the complement of left linear topologies on a ring R. Thus we consider the collection $\mathrm{C}(R)$ of those left ideals of R which do not belong to any proper left linear topology on R. Two extremes when $\mathrm{C}(R)$ consists of all proper left ideals of R and $\mathrm{C}(R) = \{0\}$ are examined.

0. Preliminaries. Let R be a ring with identity. We denote by $\mathscr{L}(R)$ the set of all left ideals of R, and by R-mod the category of all unital left R-modules. For $A \in \mathscr{L}(R)$ and a subset F of R, we set $AF^{-1} = \{x \in R \mid xF \subseteq A\}$. A nonempty subset $\mathscr{L}$ of $\mathscr{L}(R)$ is called a *left linear topology* if the following conditions are satisfied :

T1. If $I \in \mathscr{L}$, $J \in \mathscr{L}(R)$ and $I \leq J$, then $J \in \mathscr{L}$.

T2. If I and J belong to $\mathscr{L}$, then $I \cap J \in \mathscr{L}$.

T3. If $I \in \mathscr{L}$ and $a \in R$, then $Ia^{-1} \in \mathscr{L}$.

A left linear topology $\mathscr{L}$ on R is called a *left Gabriel topology* if $\mathscr{L}$ satisfies a further condition :

T4. If $I \in \mathscr{L}(R)$ and there exists $J \in \mathscr{L}$ such that $Ij^{-1} \in \mathscr{L}$ for every $j \in J$, then $I \in \mathscr{L}$.

A left linear topology $\mathscr{L}$ is called *proper* if $0 \notin \mathscr{L}$. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are left linear topologies on R, we define $\mathscr{L}_1 \leq \mathscr{L}_2$ if every member of $\mathscr{L}_1$ is a member of $\mathscr{L}_2$. A subclass $\mathscr{T}$ of R-mod is called a *hereditary pretorsion class* if $\mathscr{T}$ is closed under isomorphisms, submodules, factor modules and direct sums. $\mathscr{T}$ is called *proper* if $R \notin \mathscr{T}$. A preradical r for R-mod is called *left exact* if $r(N) = r(M) \cap N$ for every $M \in R$-mod and every submodule N of M. It is called *proper* if $r(R) \neq R$, and is called *cofaithful* if $r(Q) = Q$ for every injective $Q \in R$-mod. For preradicals r and s for

R-mod, we define a preradical $r+s$ by $(r+s)(M) = r(M)+s(M)$ for all $M \in R$-mod. For a module $Q \in R$-mod, we define a preradical t_Q for R-mod by $t_Q(M) = \sum \mathrm{Im}\alpha$, α ranging over $\mathrm{Hom}_R(Q, M)$, for each $M \in R$-mod. We remark that t_Q is cofaithful if and only if Q is cofaithful, i.e., Q generates all injective left R-modules, or equivalently, R can be embedded in a finite direct sum of copies of Q ([1, Proposition 4. 5. 4]). We naturally define the ordering of hereditary pretorsion classes of R-mod and that of left exact preradicals for R-mod. It is well known that there is an order preserving bijective correspondence between left linear topologies on R, hereditary pretorsion classes of R-mod and left exact preradicals for R-mod (see [9, p. 145]).

1. Maximal linear topologies. It is not assured that, for a given proper left Gabriel topology on R, there exists a maximal left Gabriel topology containing given one. In [7, Theorem 3. 4], Meijer and Smith proved that the above property on a ring R holds if and only if every nonzero injective left R-module has a nonzero submodule whose annihilator is an M-ideal. If R satisfies the maximum condition for ideals, then R has the above property ([3, Proposition 3. 2]). But we can prove the next

Proposition 1.1. *For every proper left linear topology $\mathscr{L}$ on R, there exists a maximal left linear topology containing $\mathscr{L}$.*

Proof. This is done by Zorn's lemma.

Lemma 1.2. *For every left linear topology $\mathscr{L}$ on R and left ideal A of R, there exists a unique minimal left linear topology $\mathscr{L}^*$ containing $\mathscr{L}$ and A. For $J \in \mathscr{L}(R)$, J belongs to $\mathscr{L}^*$ if and only if there exist $I \in \mathscr{L}$ and a finite subset F of R such that $J \geq I \cap AF^{-1}$.*

Proof. Let $\mathscr{L}^*$ be the set of left ideals J of R such that there exist $I \in \mathscr{L}$ and a finite subset F of R satisfying $J \geq I \cap AF^{-1}$. It is sufficient to show that $\mathscr{L}^*$ is in fact a left linear topology. Clearly $\mathscr{L}^*$ satisfies T1. Assume J_1 and J_2 belong to $\mathscr{L}^*$. Then there exist left ideals I_1 and I_2 and finite subsets F_1 and F_2 of R such that $J_i \geq I_i \cap AF_i^{-1}$ $(i = 1, 2)$. Since $I_1 \cap I_2 \in \mathscr{L}$ and $AF_1^{-1} \cap AF_2^{-1} = A(F_1 \cup F_2)^{-1}$, we have $J_1 \cap J_2 \geq (I_1 \cap I_2) \cap A(F_1 \cup F_2)^{-1}$, proving $\mathscr{L}^*$ satisfies T2. Now assume $J \in \mathscr{L}^*$ and $a \in R$. Then there exist $I \in \mathscr{L}$ and a finite subset F of R such that $J \geq I \cap AF^{-1}$. Now we have $Ja^{-1} \geq (I \cap AF^{-1})a^{-1} = Ia^{-1} \cap (AF^{-1})a^{-1}$

$= Ia^{-1} \cap A(aF)^{-1}$. Since $Ia^{-1} \in \mathscr{L}$, we obtain $Ja^{-1} \in \mathscr{L}^*$, proving $\mathscr{L}^*$ satisfies T3.

Now we have a criterion of the maximality of left linear topologies.

Theorem 1.3. *The following conditions are equivalent for a proper left linear topology $\mathscr{L}$ on R :*

(1) *$\mathscr{L}$ is maximal.*

(2) *For each left ideal $A \notin \mathscr{L}$, there exist $I \in \mathscr{L}$ and a finite subset F of R such that $I \cap AF^{-1} = 0$.*

(3) *For each left ideal $A \notin \mathscr{L}$, there exist $I \in \mathscr{L}$ and a natural number n such that R can be embedded in $R/I \oplus (R/A)^{(n)}$.*

(4) *For each left ideal $A \notin \mathscr{L}$, there exists $I \in \mathscr{L}$ such that $R/I \oplus R/A$ is cofaithful.*

Proof. $\mathscr{L}$ is maximal if and only if, for each left ideal $A \notin \mathscr{L}$, 0 belongs to the unique minimal left linear topology containing $\mathscr{L}$ and A. Hence by using Lemma 1.2, we have (1)$\Leftrightarrow$(2). Now assume, for each left ideal $A \notin \mathscr{L}$, there exist $I \in \mathscr{L}$ and a finite subset $\{r_1, \cdots, r_n\}$ of R such that $I \cap Ar_1^{-1} \cap \cdots \cap Ar_n^{-1} = 0$. Then R is embedded in $R/I \oplus R/Ar_1^{-1} \oplus \cdots \oplus R/Ar_n^{-1}$. But $R/Ar_i^{-1} \cong (Rr_i + A)/A \leq R/A$ for each $i = 1, \cdots, n$. Hence we have (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (4) is trivial. Finally, assume for each left ideal $A \notin \mathscr{L}$, there exist $I \in \mathscr{L}$ and a natural number n with a monomorphism $f: R \to (R/I)^{(n)} \oplus (R/A)^{(n)}$. Put $f(1) = (\bar{s}_1, \cdots, \bar{s}_n, \bar{r}_1, \cdots, \bar{r}_n)$, where $s_i, r_i \in R$ and $\bar{s}_i = s_i + I$ and $\bar{r}_i = r_i + A$ for $i = 1, \cdots, n$. Since $f(x) = (x\bar{s}_1, \cdots, x\bar{s}_n, x\bar{r}_1, \cdots, x\bar{r}_n) = 0$ implies $x = 0$, we have $Is_1^{-1} \cap \cdots \cap Is_n^{-1} \cap Ar_1^{-1} \cap \cdots \cap Ar_n^{-1} = 0$. Thus we have proved (4) $\Rightarrow$ (2), because $Is_1^{-1} \cap \cdots \cap Is_n^{-1} \in \mathscr{L}$.

Corollary 1.4. *The following conditions are equivalent for a proper hereditary pretorsion class $\mathscr{S}$ of R-mod :*

(1) *$\mathscr{S}$ is maximal.*

(2) *For each (cyclic) left R-module $M \notin \mathscr{S}$, there exist a cyclic left R-module $C \in \mathscr{S}$ and a natural number n such that R can be embedded in $C \oplus M^{(n)}$.*

(3) *For each (cyclic) left R-module $M \notin \mathscr{S}$, there exists a cyclic left R-module $C \in \mathscr{S}$ such that $C \oplus M$ is cofaithful.*

Corollary 1.5. *The following conditions are equivalent for a proper*

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left exact preradical r for R-mod :

(1) *r is maximal.*

(2) *For each (cyclic) left R-module M with $r(M) \neq M$, there exists a cyclic left R-module C with $r(C) = C$ such that $t_C + t_M$ is cofaithful.*

In [8] Rubin called a left ideal A of R *weakly essential* if $AF^{-1} \neq 0$ for every finite subset F of R. Note that, if a left ideal A is weakly essential, then AX^{-1} is also weakly essential for every finite subset X of R. We remark that every member of a proper left linear topology on R is weakly essential. Now we shall consider the case when a left linear topology is unique maximal.

Proposition 1.6. *The following conditions are equivalent for a proper left linear topology $\mathscr{L}$ on R :*

(1) *$\mathscr{L}$ is unique maximal.*

(2) *$\mathscr{L}$ coincides with the set of all weakly essential left ideals of R.*

(3) *$\mathscr{L}$ contains all weakly essential left ideals of R.*

Proof. For a left ideal A of R, we put $\mathscr{L}_A$ a unique minimal left linear topology containing A. Then $\mathscr{L}_A$ consists of those left ideals B such that $B \geq AF^{-1}$ for some finite subset F of R.

(1) $\Rightarrow$ (2). If $A \in \mathscr{L}$, then $AF^{-1} \in \mathscr{L}$ for every finite subset F of R. Since $\mathscr{L}$ is proper, we see that A is weakly essential. Conversely assume A is a weakly essential left ideal of R. Then $\mathscr{L}_A$ is proper and so $\mathscr{L}_A \subseteq \mathscr{L}$. Hence we have $A \in \mathscr{L}$.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (1). Let $\mathscr{L}'$ be a proper left linear topology on R. For each $A \in \mathscr{L}'$, we have $AF^{-1} \in \mathscr{L}'$ for every finite subset F of R. Since $\mathscr{L}'$ is proper, we see A is weakly essential, and so $A \in \mathscr{L}$ by (3). Therefore we have proved $\mathscr{L}$ is unique maximal.

The following corollary was proved by Nicholson and Sarath by using the notion of α-weak essentiality. But we can prove this directly.

Corollary 1.7 (Nicholson and Sarath [6, Theorem 1]). *The following conditions are equivalent for a ring R with the set $\mathscr{L}$ of all weakly essential left ideals of R :*

(1) *R has a unique maximal left linear topology.*

(2) *$\mathscr{L}$ forms a left linear topology.*

(3) *If A and B belong to $\mathscr{L}$, then $A \cap B \neq 0$.*

Proof. (1)$\Leftrightarrow$(2). This is clear by using Proposition 1.6.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (2). Clearly $\mathscr{L}$ satisfies T1. As noted above, $\mathscr{L}$ also satisfies T3. Now assume A and B belong to $\mathscr{L}$. For every finite subset F of R, we see that AF^{-1} and BF^{-1} belong to $\mathscr{L}$. Hence $(A \cap B)F^{-1} = AF^{-1} \cap BF^{-1} \neq 0$ by (3). Thus $A \cap B$ belongs to $\mathscr{L}$. Therefore we showed that $\mathscr{L}$ satisfies T2.

Example 1.8. Let R be a ring and $\mathscr{L}$ the set of all essential left ideals of R. It is well known that $\mathscr{L}$ is a proper left linear topology on R. By using Theorem 1.3, we notice that $\mathscr{L}$ is maximal if and only if every weakly essential left ideal of R is essential. In this case, $\mathscr{L}$ is unique maximal by Proposition 1.6. In case R is commutative, we remark that $\mathscr{L}$ is maximal if and only if every nonzero ideal of R is essential. Thus we conclude that, if R is a commutative semiprime ring, $\mathscr{L}$ is maximal if and only if R is prime.

2. The complement of linear topologies. In [7] Meijer and Smith concerned with the collection $\mathrm{N}(R)$ of those left ideals of R which do not belong to any proper left Gabriel topology on R. As mentioned in [7, Lemma 2.1], a left ideal I belongs to $\mathrm{N}(R)$ if and only if $\mathrm{Hom}_R(R/I, E) \neq 0$ for every nonzero injective left R-module E. Now we shall consider the set

$$\mathrm{C}(R) = \{I \in \mathscr{L}(R) \mid I \notin \mathscr{L} \text{ for every proper left linear topology } \mathscr{L} \text{ on } R\}.$$

Clearly $0 \in \mathrm{C}(R)$ and $R \notin \mathrm{C}(R)$. If $A \in \mathscr{L}(R)$ and $A \leq B$ for some $B \in \mathrm{C}(R)$, then $A \in \mathrm{C}(R)$. Remark that $\mathrm{C}(R) \subseteq \mathrm{N}(R)$.

Theorem 2.1. *The following statements are equivalent for a left ideal A of a ring R:*

(1) $A \in \mathrm{C}(R)$.

(2) *A is not weakly essential, i.e., $AF^{-1} = 0$ for some finite subset F of R.*

(3) *R/A is cofaithful.*

Proof. (1)$\Leftrightarrow$(2). For a left ideal A, $A \in \mathrm{C}(R)$ if and only if $0 \in \mathscr{L}$ for every left linear topology $\mathscr{L}$ containing A, or equivalently

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0 belongs to the unique minimal left linear topology containing A. As noted in the proof of Proposition 1.6, $0 \in \mathscr{L}_A$ if and only if $AF^{-1} = 0$ for some finite subset F of R.

(2)⇔(3). This is proved by the same method as is used in the proof of Theorem 1.3. (See [1, Proposition 4.5.4]).

A left linear topology $\mathscr{L}$ is called *super* if $\mathscr{L}$ contains a unique minimal member. Such a member is in fact a two-sided ideal. We denote by $\mathrm{C}_s(R)$ the set of those left ideals which do not belong to any proper super left linear topology on R. Clearly $\mathrm{C}_s(R) \supseteq \mathrm{C}(R)$. If R is left artinian, then every left linear topology on R is super, and so $\mathrm{C}_s(R) = \mathrm{C}(R)$. For a ring R with Jacobson radical J, it was proved in [7, Proposition 2.9] that $\mathrm{N}(R)$ consists of all proper left ideals of R if and only if J is right T-nilpotent and R/J is a simple artinian ring. By the definition, $\mathrm{C}_s(R)$ consists of all proper left ideals of R if and only if R is a two-sided simple ring.

Theorem 2.2. *The following statements are equivalent for a ring R :*
(1) $\mathrm{C}(R)$ *contains all maximal left ideals of* R.
(2) $\mathrm{C}(R)$ *consists of all proper left ideals of* R.
(3) R *is a simple artinian ring.*

Proof. (1) ⇒ (2). Clear.

(2) ⇒ (3). Assume (2). Then every nonzero cyclic left R-module is cofaithful by Theorem 2.1. Thus every nonzero left R-module is also cofaithful. In particular every nonzero left ideal of R is cofaithful, and so R is left strongly prime (see [5, Proposition 2.5]). Also since every faithful left R-module is cofaithful, $\mathrm{soc}({}_RR) \neq 0$ by [2, Proposition 1]. Hence R must be simple artinian by [5, Theorem 4.3].

(3) ⇒ (1). Assume I is a maximal left ideal of R. Then ${}_R(R/I)$ is cofaithful and so I belongs to $\mathrm{C}(R)$ by Theorem 2.1.

In [7] the other extreme when $\mathrm{N}(R) = \{0\}$ was considered. It was shown in [7, Theorem 6.4] that $\mathrm{N}(R) = \{0\}$ if and only if R is a reduced ring and $Ra + 0a^{-1}$ is essential left ideal of R for all $a \in R$. We remark that $\mathrm{C}_s(R) = \{0\}$ if and only if every nonzero left ideal of R contains a nonzero ideal of R.

Theorem 2.3. *The following statements are equivalent for a ring R :*
(1) $\mathrm{C}(R) = \{0\}$.

(2) *Every nonzero left ideal of R is weakly essential.*

(3) *Every nonzero cyclic left ideal of R is weakly essential.*

(4) *For every nonzero element a of R and elements $r_1,\cdots,r_n$ of R, there exist a nonzero element a' of R and elements r_i' in R such that $a'r_i = r_i'a$ $(i=1,\cdots,n)$.*

Proof. (1)$\Leftrightarrow$(2). This is clear by Theorem 2.1.

(2)$\Leftrightarrow$(3). Clear.

(3)$\Leftrightarrow$(4). Let $A = Ra$ be a nonzero cyclic left ideal of R. Then A is weakly essential if and only if, for every elements $r_1,\cdots,r_n$ of R, $Ar_1^{-1}\cap\cdots\cap Ar_n^{-1}\neq 0$ holds. This occurs if and only if, for every elements $r_1,\cdots,r_n$ of R, there exists a nonzero element a' of R such that $a'r_i\in A = Ra$ $(i=1,\cdots,n)$.

Corollary 2.4 (cf. [7, Corollaries 5.2 and 6.5]). *If R is a domain, then $\mathrm{C}(R)=\{0\}$ if and only if R satisfies the left Ore condition.*

Proof. Assume R is a left Ore domain with a classical left quotient ring $Q_{cl}^{l}(R)$. For every nonzero element a of R and elements $r_1,\cdots,r_n$ of R, there exist nonzero elements a_i' of R and elements s_i of R such that $r_ia^{-1} = a_i'^{-1}s_i$ $(i=1,\cdots,n)$. As is well known (see [4, p. 392]), there exist a nonzero element a' of R and elements t_i of R such that $a_i'^{-1} = a'^{-1}t_i$ $(i=1,\cdots,n)$. Put $r_i' = t_is_i$ $(i=1,\cdots,n)$. Thus we have $a'r_i = r_i'a$ $(i=1,\cdots,n)$, and so $\mathrm{C}(R)=\{0\}$. We can also show this fact by using [7, Corollary 5.2] with $\mathrm{C}(R)\subseteq \mathbf{N}(R)$. The reverse implication is obvious.

Remark 2.5. The property that $\mathrm{C}(R)=\{0\}$ of rings R is not a Morita invariant. To see this, let K be a field. By Theorem 2.3, we see $\mathrm{C}(K)=\{0\}$. But consider the ring R of $n\times n$ matrices over K for some $n>1$. As shown in Theorem 2.2, we have $\mathrm{C}(R)\neq\{0\}$. On the other hand, the property that R has a unique maximal left linear topology is a Morita invariant ([6, Corollary to Theorem 2]). Hence we conclude that the above two properties on R are not equivalent.

By using Theorem 2.3, we shall prove the next two propositions.

Proposition 2.6. *If R is a left order in a ring Q, then $\mathrm{C}(Q)=\{0\}$ implies $\mathrm{C}(R)=\{0\}$. Furthermore, if R is a domain, then $\mathrm{C}(Q)=\{0\}$.*

Proof. Suppose there are given elements $r(\neq 0)$, $r_1,\cdots,r_n$ in R.

By $C(Q) = \{0\}$, there exist elements $q(\neq 0)$, $q_1, \cdots, q_n$ in Q such that $qr_i = q_i r$ $(i = 1, \cdots, n)$. We can find a regular element r' in R with $r'q(\neq 0)$, $r'q_1, \cdots, r'q_n \in R$. Thus we have $(r'q)r_i = (r'q_i)r$ $(i = 1, \cdots, n)$, and so $C(R) = \{0\}$.

Now assume R is a domain. For every elements $q(\neq 0)$, $q_1, \cdots, q_n$ of Q, there exist a regular element r in R with $rq(\neq 0)$, $rq_1, \cdots, rq_n \in R$. Since $C(R) = \{0\}$ by Corollary 2.4, there exist $r'(\neq 0)$, $r_1', \cdots, r_n'$ in R such that $r'(rq_i) = r_i'(rq)$ $(i = 1, \cdots, n)$. Noting that $r'r(\neq 0)$ and $r_i'r$ $(i = 1, \cdots, n)$ belong to Q, we obtain $C(Q) = \{0\}$.

Proposition 2.7. *Suppose $R = R_1 \times \cdots \times R_n$ is a direct sum of rings R_i $(i = 1, \cdots, n)$. Then $C(R) = \{0\}$ if and only if $C(R_i) = \{0\}$ for all $i = 1, \cdots, n$.*

Proof. We may assume $n = 2$. Let S and T be rings. Assume $C(S) = \{0\}$ and $C(T) = \{0\}$. Let $(s, t)(\neq 0)$, $(s_1, t_1), \cdots, (s_n, t_n)$ be elements of $S \times T$. We may assume that $s \neq 0$. By $C(S) = \{0\}$, there exist $s'(\neq 0)$, $s_1', \cdots, s_n'$ in S such that $s's_i = s_i's$ $(i = 1, \cdots, n)$. If $t = 0$, then we have $(s', t)(s_i, t_i) = (s_i', t_i)(s, t)$ $(i = 1, \cdots, n)$. If $t \neq 0$, by $C(T) = \{0\}$, there exist $t'(\neq 0)$, $t_1', \cdots, t_n'$ in T such that $t't_i = t_i't$ $(i = 1, \cdots, n)$, and so we have $(s', t')(s_i, t_i) = (s_i', t_i')(s, t)$ $(i = 1, \cdots, n)$. Therefore we have $C(S \times T) = \{0\}$.

Conversely assume $C(S \times T) = \{0\}$. To show $C(S) = \{0\}$, let $s(\neq 0)$, $s_1, \cdots, s_n$ be elements of S. For the elements $(s, 0)$, $(s_1, 1), \cdots, (s_n, 1)$ in $S \times T$, there exist elements $(s', t')(\neq 0)$, $(s_1', t_1'), \cdots, (s_n', t_n')$ in $S \times T$ such that $(s', t')(s_i, 1) = (s_i', t_i')(s, 0)$ $(i = 1, \cdots, n)$. Then we have $s's_i = s_i's$ $(i = 1, \cdots, n)$ and $s' \neq 0$ because $t' = 0$. Therefore we showed $C(S) = \{0\}$.

Example 2.8. There may be many rings R such that $C(R)$ are not extreme. To give such an example, we shall calculate $C(R)$ where R is the 2×2 upper triangular matrix ring over a field K. There are three types of minimal left ideals of R, namely $A = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $B = \left\{ \begin{pmatrix} xa & xb \\ 0 & 0 \end{pmatrix} \middle| \; x \in K \right\}$ for some fixed nonzero elements a and b of K. Let e_{11}, e_{12} and e_{22} be matrix units in R. Since $Ae_{12}^{-1} \cap Ae_{22}^{-1} = 0$, A belongs to $C(R)$. Also since $Be_{11}^{-1} \cap Be_{22}^{-1} = 0$, B belongs to $C(R)$. But since C is an ideal of R, it is weakly essential and so C does not belong to $C(R)$. Now let I be a left ideal of R which contains A or some B strictly. Then I also contains C and so I does not belong to $\mathrm{C}(R)$. Thus we conclude that $\mathrm{C}(R)$ consists precisely of A and those left ideals B.

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